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Unification based on $SO^*(14)$ Yang–Mills theory: the gauge field Lagrangian

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Abstract. Gravity can be described as a gauge field theory where connection and curvature are $so(2,3)$ valued. In the standard gauge field theory for strong and electroweak interaction, corresponding quantities take their value in the $su(3)\oplus su(2)\oplus u(1)$ algebra. Therefore, unification of gravity with the other fundamental interactions is obtained by using the non-compact simple real Lie algebra $so^*(14)\supset so(2,3)\oplus su(3)\oplus su(2)\oplus u(1)$ as a unifying algebra.

Commutation relations for $so^*(14)$ are derived in a basis adapted to this subalgebra structure. The $so^*(14)$ gauge field defined by a connection one-form on the $SO^*(14)$ principal fibre bundle unifies the fundamental interactions in particle physics, gravity included. The 91 components of the connection contain the 10 anti-de Sitter gauge fields, the 12 gauge bosons associated with $SU(3)\oplus SU(2)\oplus U(1)$, two $SU(3)$ triplets of lepto-quark bosons, an anti-de Sitter five-vector which is also an $SU(2)$ triplet and finally two $SU(3)$ triplets of four-spinors which are also $SU(2)$ doublets. Although $so^*(14)$ is a Lie algebra and not a superalgebra, it is a general property of the theory that bosons and fermions can be incorporated in irreducible supermultiplets.

The unified gauge field Lagrangian is defined by the Yang–Mills Weil form on the $SO^*(14)$ principal bundle.

1. Introduction

The simple real Lie algebra $so^*(14)$ is non-compact [1], and thus the finite-dimensional irreducible representations are non-Hermitian. Nevertheless, it is physically interesting to consider this algebra in the context of a gauge theory describing the interactions between elementary particles. The reason was given in [2]. Since $so^*(14)\supset so(2,3)\oplus su(3)\oplus su(2)\oplus u(1)$, there is the possibility to combine the subalgebra $su(3)\oplus su(2)\oplus u(1)$ on which the standard theory of strong and electroweak interactions is based, with the Lorentz algebra $so(1,3)\subset so(2,3)$.

The Lie group $SO^*(14)$ is not a group of bosonic symmetries. The symmetry connects particles of integer spin with particles of half-odd-integer spin. In general, an irrep of $so^*(14)$ will contain both tensor and spinor representations of $so(1,3)$. Bosons and fermions are incorporated in irreducible supermultiplets, although $so^*(14)$ is a Lie algebra and not a superalgebra. Therefore, a gauge theory based on $SO^*(14)$ escapes the 'no-go' theorem of Coleman and Mandula, who showed on very general assumptions that any group of bosonic symmetries of the S-matrix in relativistic field theory is the

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direct product of the Poincaré group with an internal symmetry group [3]. Moreover, in the model presented here, spacetime is never perfectly Minkowskian but anti-de Sitter, and the spacetime isometry group is the anti-de Sitter group $SO(2,3)$ instead of the Poincaré group. It is only in the limit in which the constant curvature of the anti-de Sitter space goes to zero that this space tends to Minkowski space and the isometry group $SO(2,3)$ contracts to the Poincaré group (e.g. see [4]). The fact that the $SO^*(14)$ symmetry has the Poincaré group as spacetime symmetry only as a limiting case also implies that the theorem of O'Raifeartaigh [5] does not apply, such that all members of an irreducible multiplet of the internal symmetry group are allowed to have different masses. It is for this reason that Tait and Cornwell [6] considered all unifications of the de Sitter algebras with a real simple internal symmetry algebra. However, non-trivial unifications of anti-de Sitter with the symmetries of the standard model are only possible if the condition for a simple internal symmetry is relaxed to the case where it is neither simple nor semi-simple, as in the present model.

By considering the spacetime symmetry as a local symmetry, gravitational interaction is introduced. The fact that the anti-de Sitter group can be used, instead of the Poincaré group as the local symmetry on which gravity is based, has been known for some time. The formulation of gravity as a de Sitter or anti-de Sitter gauge theory started with the papers of Townsend, MacDowell and Mansouri [7]. More recently, Zardecki used the notion of the connection of Cartan, to describe gravity in such gauge formalism, and formulated the BRST invariance of the theory [8]. Gotzes and Hirshfeld used a Clifford algebra valued Cartan connection to give a geometrical formulation of anti-de Sitter gravity [9].

Essential in the formulation of these gauge theories for gravity is the notion of fibre bundle reduction and the related concept of symmetry breaking [10–12]. If $P(M,G)$ is a principal fibre bundle with structure group G over spacetime M and H a closed subgroup of G, then the existence of a principal subbundle $Q(M,H)$, is equivalent to the existence of a section (a 'physical Higgs field') $\psi: M \to P/H$, where P/H is the associated bundle to P by the action of G on the coset space G/H. There exists a one-to-one correspondence between these sections and equivariant mappings $\phi: P \to G/H \subset V$ of the type (ρ, V) where V is the vector space on which G acts through a representation $\rho: G \to GL(V)$ and G/H is the orbit space $\rho(G)\cdot v_0$, with $v_0 \in G/H$ an H-fixed point in V. Fulp and Norris [12] refer to ϕ as a 'symmetry breaking Higgs field'. In fact $Q=\phi^{-1}(v_0)$. Using this concept of symmetry breaking, the original Lagrangian defined in the G-principal bundle P can be re-expressed in terms of quantities defined in the H-principal sub-bundle Q. In [13] it was shown that within this geometric framework it is possible to describe gravity and the electroweak theory with one type of Lagrangian, a Yang–Mills Weil form on a principal fibre bundle over spacetime whereby the curvature two-form takes its values in the anti-de Sitter and $su(2)\oplus u(1)$ algebra, respectively. Having described gravity as a gauge theory with symmetry breaking, geometrically seen completely similar to electroweak theory, unification of both interactions within a Yang–Mills gauge theory becomes meaningful. The unifying algebra is then $so^*(10) \supset so(2,3)\oplus su(2)\oplus u(1)$. It is more natural to also incorporate the strong interaction in this unification. A unifying algebra is then $so^*(14)$, and we have to consider a symmetry breaking $SO^*(14) \to SO(2,3)\otimes SU(3)\otimes SU(2)\otimes U(1)$.

If our unifying gauge group is to be a 'good' symmetry it should have a single gauge coupling at super-high energies so that all interactions are truly unified. It should therefore be a simple group. A natural choice is then $SO^*(14)$ since it is the smallest simple Lie group that contains the required subsymmetries. It is, however, possible to consider other gauge groups as a unifying group. In fact,

$$so(2,3)\oplus su(3)\oplus su(2)\oplus u(1)\subset so^*(6)\oplus so^*(8)\subset so^*(14)\subset so^*(16) \tag{1.1}$$

and $so^*(16)$ is a maximum subalgebra of a non-compact real form of E_8.

It is important to notice that undesirable consequences connected with the use of a non-compact real form as a gauge symmetry, such as negative probabilities and non-unitary S-matrix at the quantum level, can be eliminated. This was shown by Margolin and Strazhev [14], who performed Yang-Mills field quantization in BRST formalism for non-compact gauge groups.

It is the purpose of this paper to describe a unified gauge field for electroweak, strong and gravitational interaction. Therefore, after defining the Lie algebra $so^*(2n)$ in section 2, we define in section 3 the basis operators for $so^*(14)$ appropriate to the reduction scheme. Their commutation relations are given in section 4. In section 5, the $so^*(14)$-valued connection one-form is introduced and its curvature form calculated. In section 6 the unifying gauge field Lagrangian is defined and elaborated. A discussion and interpretation of the theory presented can be found in section 7.

2. The Lie algebra $so^*(2n)$

The Lie algebra $so^*(2n)$ is a real form of D_n, the complex extension of $so(2n)$. For $n>2$, $so^*(2n)$ is simple and for $n>1$, these algebras are non-compact.

Let E_{vw} be the $n\times n$ matrix with all entries zero except on the intersection of the vth row and wth column where it has the entry 1, and define

$$A_{vw}=E_{vw}-E_{wv} \qquad S_{vw}=\mathrm{i}(E_{vw}+E_{wv}) \qquad v,w=1,2,\ldots,n \tag{2.1}$$

$$\begin{aligned} l_{vw}&=-l_{wv}=-\mathrm{i}\begin{bmatrix}A_{vw}&0\\0&A_{vw}\end{bmatrix} \qquad & m_{vw}&=m_{wv}=-\mathrm{i}\begin{bmatrix}S_{vw}&0\\0&-S_{vw}\end{bmatrix}\\ p_{vw}&=-p_{wv}=-\mathrm{i}\begin{bmatrix}0&A_{vw}\\A_{wv}&0\end{bmatrix} \qquad & q_{vw}&=-q_{wv}=\begin{bmatrix}0&A_{vw}\\A_{vw}&0\end{bmatrix}. \end{aligned} \tag{2.2}$$

The matrices $\mathrm{i}l_{vw}$, $\mathrm{i}p_{vw}$, $\mathrm{i}q_{vw}$ $(v>w)$ and $\mathrm{i}m_{vw}$ $(v\geqslant w)$ provide a standard basis for $so^*(2n)$. We will follow the physical convention of putting an 'i' on the right-hand side of commutation relations. The fundamental commutation relations of $so^*(2n)$ are then given by [2]

$$\begin{aligned}
[m_{vw},m_{tu}]&=\mathrm{i}(\delta_{wt}l_{vu}-\delta_{vt}l_{uw}-\delta_{vu}l_{tw}+\delta_{wu}l_{vt})\\
[m_{vw},p_{tu}]&=\mathrm{i}(-\delta_{wt}q_{vu}+\delta_{vt}q_{uw}-\delta_{vu}q_{tw}+\delta_{wu}q_{vt})\\
[m_{vw},q_{tu}]&=\mathrm{i}(\delta_{wt}p_{vu}-\delta_{vt}p_{uw}+\delta_{vu}p_{tw}-\delta_{wu}p_{vt})\\
[p_{vw},p_{tu}]&=\mathrm{i}(\delta_{wt}l_{vu}+\delta_{vt}l_{uw}-\delta_{vu}l_{tw}-\delta_{wu}l_{vt})=[q_{vw},q_{tu}]\\
[p_{vw},q_{tu}]&=\mathrm{i}(-\delta_{wt}m_{vu}+\delta_{vt}m_{uw}-\delta_{vu}m_{tw}+\delta_{wu}m_{vt})\\
[l_{vw},x_{tu}]&=\mathrm{i}(-\delta_{wt}x_{vu}+\delta_{vt}x_{wu}+\delta_{vu}x_{tw}-\delta_{wu}x_{tv})
\end{aligned} \tag{2.3}$$

where x_{tu} denotes any of the basis operators l_{tu}, m_{tu}, p_{tu}, q_{tu}.

3. Basis operators

Since we consider a symmetry breaking $G=SO^*(14)\to H=SO(2,3)\otimes SU(3)\otimes SU(2)\otimes U(1)$ and a corresponding fibre bundle reduction from $P(M,G)$ to $Q(M,H)$, we define here new basis operators for $so^*(14)$ appropriate to such a reduction scheme. The Lie algebra $\mathscr{G}=so^*(14)$ is written as a vector space direct sum.

$$\mathscr{G}=\mathscr{H}\oplus\mathscr{T}=so(2,3)\oplus su(3)\oplus su(2)\oplus u(1)\oplus\mathscr{T} \tag{3.1}$$

where $\mathscr{H}$ is the Lie algebra of H and $\mathscr{T}$ the complementary vector subspace of $\mathscr{G}$. Basis operators for the anti-de Sitter algebra $so(2,3)$, for the $su(3)$, $su(2)$ and $u(1)$ algebras, are denoted as $\{K_{AB}=-K_{BA}\}$ $(A,B=0,1,2,3,4)$, $\{T_k\}$ $(k=1,\ldots,8)$, $\{t_s\}$ $(s=1,2,3)$, and Y, respectively. They are defined by

$$\begin{aligned}
&K_{01}=\tfrac{1}{2}(p_{13}-p_{24}) && K_{02}=-\tfrac{1}{2}(q_{13}+q_{24}) && K_{03}=-\tfrac{1}{2}(p_{23}+p_{14})\\
&K_{12}=-\tfrac{1}{4}(m_{11}-m_{22}+m_{33}-m_{44}) && K_{13}=\tfrac{1}{2}(l_{12}+l_{34}) && K_{23}=-\tfrac{1}{2}(m_{12}+m_{34})\\
&K_{40}=+\tfrac{1}{4}(m_{11}+m_{22}+m_{33}+m_{44}) && K_{41}=\tfrac{1}{2}(q_{13}-q_{24}) && K_{42}=+\tfrac{1}{2}(p_{13}+p_{24})\\
&K_{43}=-\tfrac{1}{2}(q_{23}+q_{14})
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&T_1=\tfrac{1}{2}m_{56} && T_2=\tfrac{1}{2}l_{56} && T_3=\tfrac{1}{4}(m_{55}-m_{66}) && T_4=\tfrac{1}{2}m_{57}\\
&T_5=\tfrac{1}{2}l_{57} && T_6=\tfrac{1}{2}m_{67} && T_7=\tfrac{1}{2}l_{67} && T_8=\left(\frac{1}{4\sqrt{3}}\right)(m_{55}+m_{66}-2m_{77})
\end{aligned} \tag{3.3}$$

$$t_1=-\tfrac{1}{2}(m_{13}+m_{24}) \qquad t_2=-\tfrac{1}{2}(l_{13}+l_{24}) \qquad t_3=\tfrac{1}{4}(m_{11}+m_{22}-m_{33}-m_{44}) \tag{3.4}$$

$$Y=-\tfrac{1}{3}(m_{55}+m_{66}+m_{77}). \tag{3.5}$$

Basis operators for the complement $\mathscr{T}$ of $\mathscr{H}$ in $\mathscr{G}$, are defined by $\{S_{As}\}$, $\{S_\mu, \bar{S}^\mu\}$ $(\mu=1,2,3)$ and $\{S_{\alpha\mu\sigma}, \bar{S}^{\alpha\mu\sigma}=: S^{\beta\mu\sigma}(\gamma^0)^\alpha_\beta\}$ $(\alpha,\beta=1,2,3,4;\ \sigma=1,2)$, where

$$\begin{aligned}
&S_{01}=\tfrac{1}{2}(q_{12}-q_{34}) && S_{02}=\tfrac{1}{2}(p_{12}+p_{34}) && S_{03}=\tfrac{1}{2}(q_{14}-q_{23})\\
&S_{11}=\tfrac{1}{2}(m_{14}+m_{23}) && S_{12}=\tfrac{1}{2}(l_{14}+l_{23}) && S_{13}=-\tfrac{1}{2}(m_{12}-m_{34})\\
&S_{21}=-\tfrac{1}{2}(l_{23}-l_{14}) && S_{22}=-\tfrac{1}{2}(m_{14}-m_{23}) && S_{23}=-\tfrac{1}{2}(l_{12}-l_{34})\\
&S_{31}=\tfrac{1}{2}(m_{13}-m_{24}) && S_{32}=\tfrac{1}{2}(l_{13}-l_{24}) && S_{33}=-\tfrac{1}{4}(m_{11}-m_{22}-m_{33}+m_{44})\\
&S_{41}=\tfrac{1}{2}(p_{34}-p_{12}) && S_{42}=\tfrac{1}{2}(q_{12}+q_{34}) && S_{43}=\tfrac{1}{2}(p_{23}-p_{14})
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&S_1=\tfrac{1}{2}(p_{67}+\mathrm{i}q_{67}) && S_2=\tfrac{1}{2}(p_{75}+\mathrm{i}q_{75}) && S_3=\tfrac{1}{2}(p_{56}+\mathrm{i}q_{56})\\
&\bar{S}^1=-\tfrac{1}{2}(p_{67}-\mathrm{i}q_{67}) && \bar{S}^2=-\tfrac{1}{2}(p_{75}-\mathrm{i}q_{75}) && \bar{S}^3=-\tfrac{1}{2}(p_{56}-\mathrm{i}q_{56})
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&S_{111}=\tfrac{1}{2}(l_{45}+\mathrm{i}m_{45}) && S_{121}=\tfrac{1}{2}(l_{46}+\mathrm{i}m_{46}) && S_{131}=\tfrac{1}{2}(l_{47}+\mathrm{i}m_{47})\\
&S_{112}=\tfrac{1}{2}(l_{25}+\mathrm{i}m_{25}) && S_{122}=\tfrac{1}{2}(l_{26}+\mathrm{i}m_{26}) && S_{132}=\tfrac{1}{2}(l_{27}+\mathrm{i}m_{27})\\
&S_{211}=-\tfrac{1}{2}(l_{35}+\mathrm{i}m_{35}) && S_{221}=-\tfrac{1}{2}(l_{36}+\mathrm{i}m_{36}) && S_{231}=-\tfrac{1}{2}(l_{37}+\mathrm{i}m_{37})\\
&S_{212}=-\tfrac{1}{2}(l_{15}+\mathrm{i}m_{15}) && S_{222}=-\tfrac{1}{2}(l_{16}+\mathrm{i}m_{16}) && S_{232}=-\tfrac{1}{2}(l_{17}+\mathrm{i}m_{17})\\
&S_{311}=\tfrac{1}{2}(p_{15}-\mathrm{i}q_{15}) && S_{321}=\tfrac{1}{2}(p_{16}-\mathrm{i}q_{16}) && S_{331}=\tfrac{1}{2}(p_{17}-\mathrm{i}q_{17})\\
&S_{312}=-\tfrac{1}{2}(p_{35}-\mathrm{i}q_{35}) && S_{322}=-\tfrac{1}{2}(p_{36}-\mathrm{i}q_{36}) && S_{332}=-\tfrac{1}{2}(p_{37}-\mathrm{i}q_{37})\\
&S_{411}=\tfrac{1}{2}(p_{25}-\mathrm{i}q_{25}) && S_{421}=\tfrac{1}{2}(p_{26}-\mathrm{i}q_{26}) && S_{431}=\tfrac{1}{2}(p_{27}-\mathrm{i}q_{27})\\
&S_{412}=-\tfrac{1}{2}(p_{45}-\mathrm{i}q_{45}) && S_{422}=-\tfrac{1}{2}(p_{46}-\mathrm{i}q_{46}) && S_{432}=-\tfrac{1}{2}(p_{47}-\mathrm{i}q_{47}).
\end{aligned} \tag{3.8}$$

The $S^{\alpha\mu\sigma}$ are obtained from the corresponding $S_{\alpha\mu\sigma}$ after changing the sign before the 'i' in the second term of these basis operators.

In a finite-dimensional representation, their hermiticity properties are

$$K^{\dagger}_{AB}=K^{AB} \qquad T^{\dagger}_{k}=T_k \qquad t^{\dagger}_{s}=t_s \qquad Y^{\dagger}=Y \tag{3.9}$$
$$S^{\dagger}_{As}=S^{A}_{s} \qquad S^{\dagger}_{\mu}=\bar{S}^{\mu} \qquad S^{\dagger}_{\alpha\mu\sigma}=\bar{S}^{\alpha\mu\sigma}$$

and $S^{\alpha\mu\sigma}$ is the 'Dirac adjoint' of $S_{\alpha\mu\sigma}$. The anti-de Sitter indices A, B are raised using $\eta^{AB}=\mathrm{diag}(-1, 1, 1, 1, -1)$.

We remark that $so^*(8)\supset so(2,3)\oplus su(2)$ is generated by $\{K_{AB}, t_s, S_{As}\}$ and $so^*(6)\supset su(3)\oplus u(1)$ by $\{T_k, Y, S_\mu, \bar{S}^\mu\}$.

4. Commutation relations

After defining basis operators for $so^*(14)$, which reflect the subalgebra structure of interest here, we define the commutation relations between these basis elements. They are obtained straightforwardly by using the fundamental commutation relations given by (2.3) and the definition of the basis operators in (3.2)–(3.8).

In the sequel, $A, B, C, D=0, 1, 2, 3, 4$ are anti-de Sitter indices. The indices $\alpha, \beta, \gamma, \delta=1, 2, 3, 4$ label entries of the matrices σ_{AB} and γ^A. The $\{\sigma_{AB}\}$ provide a set of basis operators of the fundamental spinor irrep of $so(2,3)$. Explicit definition of the σ_{AB} and gamma matrices γ^A are given in the appendix. Indices $i, j, k, l=1, \ldots, 8$ and $p, q, r, s=1, 2, 3$ enumerate basis operators of $su(3)$ and $su(2)$, respectively. The indices $\kappa, \lambda, \mu, \nu, \xi=1, 2, 3$ and $\pi, \rho, \sigma, \tau=1, 2$ label the entries of the Gell–Mann matrices λ_k and Pauli matrices σ_s, respectively.

The non-zero commutators are listed below, grouped according to their action on subspaces $\mathscr{H}$ and $\mathscr{T}$:

$$[\mathscr{H}, \mathscr{H}]\subset\mathscr{H}$$

$$[K_{AB}, K_{CD}]=\mathrm{i}(\eta_{BC}K_{AD}+\eta_{AD}K_{BC}-\eta_{AC}K_{BD}-\eta_{BD}K_{AC}) \tag{4.1}$$

$$[T_i, T_j]=\mathrm{i}f^{k}_{ij}T_k \tag{4.2}$$

$$[t_p, t_q]=\mathrm{i}\varepsilon^{r}_{pq}t_r \tag{4.3}$$

$$[\mathscr{H}, \mathscr{T}]\subset\mathscr{T}$$

$$[K_{AB}, S_{Cs}]=-\mathrm{i}\eta_{AC}S_{Bs}+\mathrm{i}\eta_{BC}S_{As} \tag{4.4}$$

$$[K_{AB}, S_{\alpha\mu\sigma}]=(\sigma_{AB})^{\beta}_{\alpha}S_{\beta\mu\sigma} \tag{4.5a}$$

$$[K_{AB}, \bar{S}^{\alpha\mu\sigma}]=-\bar{S}^{\beta\mu\sigma}(\sigma_{AB})^{\alpha}_{\beta} \tag{4.5b}$$

$$[T_k, S_\mu]=\tfrac{1}{2}(\lambda_k)^{\nu}_{\mu}S_\nu \tag{4.6a}$$

$$[T_k, \bar{S}^{\mu}]=-\tfrac{1}{2}\bar{S}^{\nu}(\lambda_k)^{\mu}_{\nu} \tag{4.6b}$$

$$[T_k, S_{\alpha\mu\sigma}]=\tfrac{1}{2}(\lambda_k)^{\nu}_{\mu}S_{\alpha\nu\sigma} \tag{4.7a}$$

$$[T_k, \bar{S}^{\alpha\mu\sigma}]=-\tfrac{1}{2}\bar{S}^{\alpha\nu\sigma}(\lambda_k)^{\mu}_{\nu} \tag{4.7b}$$

$$[t_p, S_{Aq}]=\mathrm{i}\varepsilon^{r}_{pq}S_{Ar} \tag{4.8}$$

$$[t_s, S_{\alpha\mu\sigma}]=\tfrac{1}{2}(\sigma_s)^{\tau}_{\sigma}S_{\alpha\mu\tau} \tag{4.9a}$$

$$[t_s, \bar{S}^{\alpha\mu\sigma}] = -\tfrac{1}{2}\bar{S}^{\alpha\mu\tau}(\sigma_s)^{\sigma}_{\tau} \tag{4.9b}$$

$$[Y, S_\mu] = \tfrac{4}{3}S_\mu \tag{4.10a}$$

$$[Y, \bar{S}^\mu] = -\tfrac{4}{3}\bar{S}^\mu \tag{4.10b}$$

$$[Y, S_{\alpha\mu\sigma}] = -\tfrac{2}{3}S_{\alpha\mu\sigma} \tag{4.11a}$$

$$[Y, \bar{S}^{\alpha\mu\sigma}] = \tfrac{2}{3}\bar{S}^{\alpha\mu\sigma} \tag{4.11b}$$

$$[\mathscr{T}, \mathscr{T}] \subset \mathscr{G}$$

$$[S_\mu, \bar{S}^\nu] = (\lambda_k)^\nu_\mu T_k + \delta^\nu_\mu Y \tag{4.12}$$

$$[S_{Ap}, S_{Bq}] = -\mathrm{i}\delta_{pq}K_{AB} + \mathrm{i}\eta_{AB}\varepsilon^r_{pq}t_r \tag{4.13}$$

$$[S_{As}, S_{\alpha\mu\sigma}] = -\tfrac{1}{2}(\gamma_5\gamma_A)^\beta_\alpha(\sigma_s)^\tau_\alpha S_{\beta\mu\tau} \tag{4.14a}$$

$$[S_{As}, \bar{S}^{\alpha\mu\sigma}] = \tfrac{1}{2}\bar{S}^{\beta\mu\tau}(\gamma_5\gamma_A)^\alpha_\beta(\sigma_s)^\sigma_\tau \tag{4.14b}$$

$$[S_\mu, S_{\alpha\nu\sigma}] = -\mathrm{i}\varepsilon_{\mu\nu\gamma}\varepsilon_{\sigma\tau}\bar{S}^{\beta\gamma\tau}C_{\alpha\beta} \tag{4.15a}$$

$$[\bar{S}^\mu, \bar{S}^{\alpha\nu\sigma}] = -\mathrm{i}\varepsilon^{\mu\nu\gamma}\varepsilon^{\sigma\tau}S_{\beta\gamma\tau}C^{\alpha\beta} \tag{4.15b}$$

$$[S_{\alpha\mu\sigma}, S_{\beta\nu\tau}] = -\mathrm{i}\varepsilon_{\mu\nu\gamma}\varepsilon_{\sigma\tau}\bar{S}^\gamma C_{\alpha\beta} \tag{4.16a}$$

$$[\bar{S}^{\alpha\mu\sigma}, \bar{S}^{\beta\nu\tau}] = \mathrm{i}\varepsilon^{\mu\nu\gamma}\varepsilon^{\sigma\tau}S_\gamma C^{\alpha\beta} \tag{4.16b}$$

$$[S_{\alpha\mu\sigma}, \bar{S}^{\beta\nu\tau}] = \delta^\beta_\alpha\delta^\tau_\sigma(\lambda_k)^\nu_\mu T_k + \tfrac{1}{2}\delta^\beta_\alpha\delta^\nu_\mu(\sigma^s)^\tau_\sigma t_s - \tfrac{1}{2}\delta^\beta_\alpha\delta^\nu_\mu\delta^\tau_\sigma Y$$
$$+ \tfrac{1}{2}\delta^\nu_\mu\delta^\tau_\sigma(\sigma^{AB})^\beta_\alpha K_{AB} - \tfrac{1}{2}S^\nu_\mu(\gamma^5\gamma^A)^\beta_\alpha(\sigma^s)^\tau_\sigma S_{As}. \tag{4.17}$$

The commutation relation (4.1) defines the anti-de Sitter algebra. In (4.2), the f^k_{ij} are the structure constants of $su(3)$ (for a list see, for example, [15]). The commutation relations (4.3), where $\varepsilon^3_{12}=1$, are those for the $su(2)$ algebra. In a finite-dimensional representation, the commutation relations (4.5*a*) and (4.5*b*) are each other Hermitian conjugate. The same is true for the relations (4.6), (4.7), (4.9)–(4.11) and (4.14)–(4.16). Since H is connected, $[\mathscr{H}, \mathscr{T}] \subset \mathscr{T}$ means that the homogeneous space G/H is reductive [16].

5. The $so^*(14)$ gauge field

In the theory of fibre bundles, a gauge field is introduced as a connection one-form on a principal fibre bundle [17]. Therefore, let $P(M, G)$ be a $G=SO^*(14)$ principal fibre bundle, $Q(M, H)$ a $H=SO(2,3)\otimes SU(3)\otimes SU(2)\otimes U(1)$ sub-bundle of P and $\gamma: Q\to P$ the identity injection of Q into its extension P.

If $\tilde{\mu}$ is a connection one-form on P, then the restriction μ of $\tilde{\mu}$ to Q, i.e. $\mu=\gamma^*\tilde{\mu}$, splits according to the Lie algebra structure into a $\mathscr{H}$-valued part ω and a $\mathscr{T}$-valued part ϕ:

$$\mu = \omega + \phi. \tag{5.1}$$

From section 4 we have that $[\gamma_*\mathscr{H}, \mathscr{T}] \subset \mathscr{T}$, which implies that ω is a connection one-form on Q and ϕ tensorial one-form of type (Ad H, $\mathscr{T}$) on Q [18], where Ad denotes the adjoint representation of the symmetry group.

We can expand the Lie algebra-valued connection ω and the tensorial one-form ϕ as

$$\omega = \tfrac{1}{2}\omega^{AB}\mathscr{K}_{AB} + \omega^k \mathscr{T}_k + \omega^s \ell_s + \omega^Y \mathscr{Y} \tag{5.2}$$

$$\phi = \phi^{As}\mathscr{S}_{As} + \phi^{\mu}\mathscr{S}_{\mu} + \bar{\phi}_{\mu}\bar{\mathscr{S}}^{\mu} + \phi^{\alpha\mu\sigma}\mathscr{S}_{\alpha\mu\sigma} + \bar{\phi}_{\alpha\mu\sigma}\bar{\mathscr{S}}^{\alpha\mu\sigma} \tag{5.3}$$

with $\mathscr{K}_{AB} = -\mathrm{i}K_{AB}$, $\mathscr{T}_k = -\mathrm{i}T_k$, $\ell_s = -\mathrm{i}t_s$, $\mathscr{Y} = -\mathrm{i}Y$, $\mathscr{S}_{\mu} = -\mathrm{i}S_{\mu}$, $\bar{\mathscr{S}}^{\mu} = -\mathrm{i}\bar{S}^{\mu}$, $\mathscr{S}_{As} = -\mathrm{i}S_{As}$, $\mathscr{S}_{\alpha\mu\sigma} = -\mathrm{i}S_{\alpha\mu\sigma}$, $\bar{\mathscr{S}}^{\alpha\mu\sigma} = -\mathrm{i}\bar{S}^{\alpha\mu\sigma}$. Here, $\mathscr{K}_{AB}$, $\mathscr{T}_k$, ℓ_s and $\mathscr{Y}$ form a basis for the anti-de Sitter, $su(3)$, $su(2)$ and $u(1)$ algebra, respectively. If the $SO^*(14)$ symmetry is broken by the Higgs mechanism, the tensorial fields in the $\mathscr{T}$-valued part of the connection acquire a mass, while the components of ω are the gauge fields that remain massless [12].

The transformation properties of the 91 gauge fields under $SO(2,3)\otimes SU(3)\otimes SU(2)\otimes U(1)$ are

$$\begin{aligned} 91 = {} & \underset{\omega^{AB}}{(10.1.1.0)} \oplus \underset{\phi^{As}}{(5.1.3.0)} \oplus \underset{\omega^{s}}{(1.1.3.0)} \oplus \underset{\omega^{k}}{(1.8.1.0)} \oplus \underset{\omega^{Y}}{(1.1.1.0)} \\ & \oplus \underset{\phi^{\mu}}{(1.3.1.\tfrac{4}{3})} \oplus \underset{\bar{\phi}_{\mu}}{(1.\bar{3}.1.-\tfrac{4}{3})} \oplus \underset{\phi^{\alpha\mu\sigma}}{(4.3.2.-\tfrac{2}{3})} \oplus \underset{\bar{\phi}_{\alpha\mu\sigma}}{(4.\bar{3}.2.\tfrac{2}{3})} \end{aligned} \tag{5.4}$$

where the entries $(n_1.n_2.n_3.y)$ represent the representations under the $SO(2,3)$, $SU(3)$ and $SU(2)$ subgroups and the value of the $U(1)$ generator, respectively. The normalization of the $U(1)$ generator Y has been chosen as in [2]. We identify ω^{AB}, transforming as the 10-tensor adjoint representation of $SO(2,3)$, as the components of the anti-de Sitter connection one-form. The 12 gauge bosons ω^k, ω^s and ω^Y are associated with $SU(3)\otimes SU(2)\otimes U(1)$. In addition, there are 69 massive gauge fields. The $\phi^{\alpha\mu\sigma}$ and $\bar{\phi}_{\alpha\mu\sigma}$, are four-spinors transforming as a triplet under $SU(3)$ (μ-index) and as a doublet under $SU(2)$ (σ-index). They link quarks and leptons that lie within the same multiplet of $SO^*(14)$. Being fermions, baryon and lepton number-violating interactions mediated by these 'lepto-quarks' are enormously suppressed. The ϕ^{μ} and $\bar{\phi}_{\mu}$ are colour triplet lepto-quark bosons. Finally, we have an anti-de Sitter five-vector ϕ^{As}, which transforms as a triplet under $SU(2)$. If we identify Y with the weak hypercharge operator, then we find, from the Gell-Mann–Nishijima relation $Q = t_3 + \frac{1}{2}Y$ for the electric charge generator of the $U(1)_Q$ symmetry after electroweak symmetry breaking, that the lepto-quark vector fermions $\phi^{\alpha\mu\sigma}$ have electric charge $\frac{1}{6}$ and $-\frac{5}{6}$. The $\bar{\phi}_{\alpha\mu\sigma}$ carry the opposite charge. The ϕ^{μ} and $\bar{\phi}_{\mu}$ carry charges of $\frac{2}{3}$ and $-\frac{2}{3}$. From the isotriplet ϕ^{As} we have the charge polarization combination $\phi^{A\pm} = \sqrt{\frac{1}{2}}(\phi^{A1} \mp \phi^{A2})$, $\phi^{A0} = \phi^{A3}$. The fields $\phi^{A\pm}$ then have charge $+1$ and -1, whereas the ϕ^{A3} are neutral fields. These charge assignments can be verified using $[Q, \psi] = q\psi$ for fields ψ and by using the commutation relations of section 4.

The reduction $\Delta = \gamma^*\tilde{\Delta}$ to Q of the curvature $\tilde{\Delta}$ calculated from $\tilde{\mu}$ on P, can be written as [13]

$$\Delta = \mathrm{d}\mu + \tfrac{1}{2}[\mu, \mu] = \Omega + \Phi + \Sigma \tag{5.5}$$

where

$$\Omega = \mathrm{d}\omega + \tfrac{1}{2}[\omega, \omega] \tag{5.6}$$

$$\Phi = \mathrm{d}\phi + [\omega, \phi] \tag{5.7}$$

$$\Sigma = \tfrac{1}{2}[\phi, \phi]. \tag{5.8}$$

Here, [,] denotes the exterior product of Lie algebra-valued forms.

Substituting the expressions (5.2) and (5.3) for ω and ϕ into (5.6)-(5.8), and using the commutation relations given in section 4, yields (the wedge product is assumed between forms)

$$\Omega = \tfrac{1}{2}F^{AB}\mathscr{K}_{AB} + F^{k}\mathscr{T}_{k} + F^{r}\ell_{r} + F\mathscr{Y} \tag{5.9}$$

with

$$F^{AB} = \mathrm{d}\omega^{AB} + \omega^{A}_{C}\omega^{CB} \tag{5.10}$$

$$F^{k} = \mathrm{d}\omega^{k} + \tfrac{1}{2}f^{k}_{ij}\omega^{i}\omega^{j} \tag{5.11}$$

$$F^{r} = \mathrm{d}\omega^{r} + \tfrac{1}{2}\varepsilon^{r}_{pq}\omega^{p}\omega^{q} \tag{5.12}$$

$$F = \mathrm{d}\omega^{Y} \tag{5.13}$$

$$\Phi = \Phi^{As}\mathscr{S}_{As} + \Phi^{\mu}\mathscr{S}_{\mu} + \bar{\Phi}_{\mu}\bar{\mathscr{S}}^{\mu} + \Phi^{\alpha\mu\sigma}\mathscr{S}_{\alpha\mu\sigma} + \bar{\Phi}_{\alpha\mu\sigma}\bar{\mathscr{S}}^{\alpha\mu\sigma} \tag{5.14}$$

with

$$\Phi^{As} = \mathrm{d}\phi^{As} + \omega^{A}_{C}\phi^{Cs} + \mathrm{i}\varepsilon^{s}_{qr}\omega^{q}\phi^{Ar} \tag{5.15}$$

$$\Phi^{\mu} = \mathrm{d}\phi^{\mu} - \tfrac{1}{2}\mathrm{i}\omega^{i}\phi^{\nu}(\lambda_{i})^{\mu}_{\nu} - (\tfrac{4}{3})\mathrm{i}\omega^{Y}\phi^{\mu} \tag{5.16}$$

$$\bar{\Phi}_{\mu} = \mathrm{d}\bar{\phi}_{\mu} + \tfrac{1}{2}\mathrm{i}\omega^{i}\bar{\phi}_{\nu}(\lambda_{i})^{\nu}_{\mu} + (\tfrac{4}{3})\mathrm{i}\omega^{Y}\bar{\phi}_{\mu} \tag{5.17}$$

$$\Phi^{\alpha\mu\sigma} = \mathrm{d}\phi^{\alpha\mu\sigma} - \tfrac{1}{2}\mathrm{i}\omega^{AB}\phi^{\beta\mu\sigma}(\sigma_{AB})^{\alpha}_{\beta} - \tfrac{1}{2}\mathrm{i}\omega^{i}\phi^{\alpha\nu\sigma}(\lambda_{i})^{\mu}_{\nu} - \tfrac{1}{2}\mathrm{i}\omega^{s}\phi^{\alpha\mu\tau}(\sigma_{s})^{\sigma}_{\tau} + (\tfrac{2}{3})\mathrm{i}\omega^{Y}\phi^{\alpha\mu\sigma} \tag{5.18}$$

$$\bar{\Phi}_{\alpha\mu\sigma} = \mathrm{d}\bar{\phi}_{\alpha\mu\sigma} + \tfrac{1}{2}\mathrm{i}\omega^{AB}\bar{\phi}_{\beta\mu\sigma}(\sigma_{AB})^{\beta}_{\alpha} + \tfrac{1}{2}\mathrm{i}\omega^{i}\bar{\phi}_{\alpha\nu\sigma}(\lambda_{i})^{\nu}_{\mu} + \tfrac{1}{2}\mathrm{i}\omega^{s}\bar{\phi}_{\alpha\mu\tau}(\sigma_{s})^{\tau}_{\sigma} - (\tfrac{2}{3})\mathrm{i}\omega^{Y}\bar{\phi}_{\alpha\mu\sigma} \tag{5.19}$$

and, finally

$$\begin{aligned}
\Sigma = -\tfrac{1}{2}(\phi^{A}_{s}\phi^{Bs} + \mathrm{i}\phi^{\alpha\mu\sigma}\bar{\phi}_{\beta\mu\sigma}(\sigma^{AB})^{\beta}_{\alpha})\mathscr{K}_{AB} - \mathrm{i}(\phi^{\alpha\mu\sigma}\bar{\phi}_{\alpha\nu\sigma} + \phi^{\mu}\bar{\phi}_{\nu})(\lambda_{k})^{\nu}_{\mu}\mathscr{T}_{k} \\
+\tfrac{1}{2}(\varepsilon^{s}_{qr}\phi^{Aq}\phi^{r}_{A} - \mathrm{i}\phi^{\alpha\mu\sigma}\bar{\phi}_{\alpha\mu\tau}(\sigma^{s})^{\tau}_{\sigma})\ell_{s} - \mathrm{i}(\phi^{\mu}\bar{\phi}_{\mu} - \tfrac{1}{2}\phi^{\alpha\mu\sigma}\bar{\phi}_{\alpha\mu\sigma})\mathscr{Y} \\
+\tfrac{1}{2}\mathrm{i}\phi^{\alpha\mu\sigma}\bar{\phi}_{\beta\mu\tau}(\gamma^{5}\gamma^{A})^{\beta}_{\alpha}(\sigma^{s})^{\tau}_{\sigma}\mathscr{S}_{As} \\
+\tfrac{1}{2}\varepsilon^{\mu\nu\lambda}\varepsilon^{\sigma\tau}C^{\alpha\beta}\bar{\phi}_{\alpha\mu\sigma}\bar{\phi}_{\beta\nu\tau}\mathscr{S}_{\lambda} - \tfrac{1}{2}\varepsilon_{\mu\nu\lambda}\varepsilon_{\sigma\tau}C_{\alpha\beta}\phi^{\alpha\mu\sigma}\phi^{\beta\nu\tau}\bar{S}^{\lambda} \\
-\tfrac{1}{2}(\varepsilon^{\lambda\nu\mu}\varepsilon^{\tau\sigma}C^{\beta\alpha}\bar{\phi}_{\lambda}\bar{\phi}_{\beta\nu\tau} - \tfrac{1}{2}\mathrm{i}\phi^{As}\phi^{\beta\mu\tau}(\gamma_{5}\gamma_{A})^{\alpha}_{\beta}(\sigma_{s})^{\sigma}_{\tau})\mathscr{S}_{\alpha\mu\sigma} \\
-\tfrac{1}{2}(\varepsilon_{\lambda\nu\mu}\varepsilon_{\tau\sigma}C_{\beta\alpha}\phi^{\lambda}\phi^{\beta\nu\tau} + \tfrac{1}{2}\mathrm{i}\phi^{As}\bar{\phi}_{\beta\mu\tau}(\gamma_{5}\gamma_{A})^{\beta}_{\alpha}(\sigma_{s})^{\tau}_{\sigma})\bar{\mathscr{S}}^{\alpha\mu\sigma}.
\end{aligned} \tag{5.20}$$

In (5.9), F^{AB} is the anti-de Sitter curvature two-form calculated from the anti-de Sitter connection one-form $\omega^{AB}\cdot F^{k}$, F^{r} and F are the curvature two-forms calculated from the $su(3)$, $su(2)$ and $u(1)$ connection one-forms ω^{k}, ω^{r} and ω^{Y}, respectively. In (5.14), Φ^{As}, Φ^{μ}, $\bar{\Phi}_{\mu}$, $\Phi^{\alpha\mu\sigma}$ and $\bar{\Phi}_{\alpha\mu\sigma}$ are the exterior H-covariant derivatives of ϕ^{As}, ϕ^{μ}, $\bar{\phi}_{\mu}$, $\phi^{\alpha\mu\sigma}$ and $\bar{\phi}_{\alpha\mu\sigma}$, respectively.

6. Gauge field Lagrangian

Within the fibre bundle formulation of a Yang–Mills gauge theory, the gauge field Lagrangian on $P(M, G)$ is defined as a Yang–Mills Weil form [13]

$$L_2(\tilde{\Delta}, {}^*\tilde{\Delta}) \tag{6.1}$$

where $\tilde{\Delta}$ is the curvature two-form calculated from the connection one-form $\tilde{\mu}$ on P, * is the Hodge duality transformation, and L_2 an Ad(G)-invariant Weil polynomial of degree two on the Lie algebra $\mathscr{G}$. In the case $\mathscr{G} = so^*(14)$ considered here, the restriction Δ of $\tilde{\Delta}$ to Q is a tensorial two-form on Q (see section 5). Then the restriction to Q of the Weil form itself, i.e. $L_2(\Delta, {}^*\Delta) = \gamma^* L_2(\tilde{\Delta}, {}^*\tilde{\Delta})$, projects to a unique four-form $\mathscr{L}$ on M such that [13]

$$L_2(\Delta, {}^*\Delta) = \pi^* \mathscr{L} \tag{6.2}$$

where π is the projection from Q on the base M.

The invariance of (6.1) under general gauge transformations (the vertical automorphisms of P) follows from the Ad(G) invariance of L_2 and the fact that $\tilde{\Delta}$ is a tensorial two-form. Notice that in (6.2) the four-form $\mathscr{L}$ (Lagrangian on M) is defined without using pull-backs through natural sections of P defined by a choice of a local trivialization (gauge fixing) of the principal fibre bundle.

An Ad($SO^*(14)$)-invariant Weil polynomial on $so^*(14)$ of degree two is proportional to

$$L_2(X, Y) = -\tfrac{1}{2}(\mathrm{tr}(X)\,\mathrm{tr}(Y) - \mathrm{tr}(XY)) \tag{6.3}$$

where X, $Y \in so^*(14)$. Since $\mathrm{tr}(m_{tu}) = \mathrm{tr}(l_{tu}) = \mathrm{tr}(p_{tu}) = \mathrm{tr}(q_{tu}) = 0$, and

$$\begin{aligned}
&\mathrm{tr}(m_{tu} \cdot m_{vw}) = 4\delta^{tu}_{(vw)} \\
&\mathrm{tr}(l_{tu} \cdot l_{vw}) = 4\delta^{tu}_{[vw]} \\
&\mathrm{tr}(p_{tu} \cdot p_{vw}) = -4\delta^{tu}_{[vw]} \\
&\mathrm{tr}(q_{tu} \cdot q_{vw}) = -4\delta^{tu}_{[vw]} \\
&\mathrm{tr}(x_{tu} \cdot y_{vw}) = 0 \qquad \text{if } x \neq y
\end{aligned} \tag{6.4}$$

where x_{tu}, y_{vw} denotes any of the m_{tu}, l_{tu}, p_{tu}, q_{tu}, it is straightforward to obtain that

$$\begin{aligned}
&L_2(\mathscr{K}_{AB}, \mathscr{K}_{CD}) = C_1(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}) \\
&L_2(\mathscr{T}_i, \mathscr{T}_j) = C_2\delta_{ij} \\
&L_2(\ell_r, \ell_s) = C_3\delta_{rs} \\
&L_2(\mathscr{Y}, \mathscr{Y}) = C_4 \\
&L_2(\mathscr{S}_\mu, \bar{\mathscr{S}}^\nu) = C_5\delta^\nu_\mu \\
&L_2(\mathscr{S}_{Ar}, \mathscr{S}_{Bs}) = C_6\eta_{AB}\delta_{rs} \\
&L_2(\mathscr{S}_{\alpha\mu\sigma}, \bar{\mathscr{S}}^{\beta\nu\tau}) = C_7\delta^\beta_\alpha\delta^\nu_\mu\delta^\tau_\sigma
\end{aligned} \tag{6.5}$$

with others zero. We introduced constants $C_1, C_2, \ldots, C_7$ since the Weil polynomials are defined only upon an arbitrary constant.

Using the multilinearity of L_2, we obtain

$$L_2(\Delta, {}^*\Delta)=L_2(\Omega, {}^*\Omega)+L_2(\Phi, {}^*\Phi)+L_2(\Sigma, {}^*\Sigma)+2L_2(\Omega, {}^*\Sigma)+2L_2(\Phi, {}^*\Sigma). \tag{6.6}$$

We then substitute the explicit expressions for Ω, Φ, Σ as given in (5.9), (5.14) and (5.20) into the Lagrangian (6.6), and use (6.5). For the first two terms in (6.6) we obtain

$$L_2(\Omega, {}^*\Omega)=\tfrac{1}{2}C_1F^{AB*}F_{AB}+C_2F^{k*}F_k+C_3F^{s*}F_s+C_4F^*F \tag{6.7}$$

$$L_2(\Phi, {}^*\Phi)=2C_5\Phi^{\mu*}\bar{\Phi}_\mu+C_6\Phi^{As*}\Phi_{As}+2C_7\Phi^{\alpha\mu\sigma*}\bar{\Phi}_{\alpha\mu\sigma}. \tag{6.8}$$

The gauge field Lagrangian for the massless gauge fields ω^{AB}, ω^k, ω^s and ω^Y is given by (6.7). For the massive fields, the kinetic terms are given by (6.8). The last three terms in (6.6) contain the interaction terms between the massless and massive fields and self-interaction terms of the massive fields. As for the first two terms in (6.6), it is straightforward to obtain their explicit expressions.

7. Discussion

By considering a field theory based on $so^*(14)$, we are led to a unification of gravity with the other fundamental interactions according to a GUT scheme. We have started to build such a field theory by giving a field theoretical interpretation of the 91-dimensional adjoint irrep of $so^*(14)$ with respect to the reduction $so^*(14)\supset so(2,3)\oplus su(3)\oplus su(2)\oplus u(1)$. We thus obtained the pure gauge field sector. In [2], branching rules of the foregoing type are also given for the two fundamental spinor irreps, each of dimension 64, and for the defining 14-vector irrep of $so^*(14)$. Including matter fields (fermions, Higgs fields) in the theory consists of giving a field theoretical interpretation of these branchings. Although we concentrate here on the gauge field sector, we wish to note that fermions of the first generation can be placed in a pair of 64-dimensional complex spinor irreps $(64)_-$ and $(64)_+$ of $so^*(14)$. The $su(2)$ doublet of four-spinors which is an $su(3)$ triplet in the $(64)_-$ can be associated with a left-handed (u, d) quark doublet, while the $su(2)$ doublet of four spinors which is an $su(3)$ singlet in the same irrep can be associated with a left-handed lepton doublet (ν_e, e^-). Their right-handed antiparticles can be placed in the $(64)_+$. The bosonic part of these supermultiplets contain a coloured as well as a non-coloured $so(2,3)$ vector and scalar.

Gravitational interaction is created when the anti-de Sitter symmetry is broken down to a Lorentz subsymmetry. The anti-de Sitter connection $\mu'=\frac{1}{2}\omega^{AB}\mathscr{K}_{AB}$, then has to be interpreted as a Cartan connection on the anti-de Sitter frame bundle. The anti-de Sitter algebra $so(2,3)$ splits into a Lorentz subalgebra $so(1,3)$ and a complementary vector space $\mathbb{R}^{1,3}$ isomorphic with Minkowski space. The 10 generators $\mathscr{K}_{AB}$ of the anti-de Sitter group split into six generators $J_{ab}=:\mathscr{K}_{ab}$ of the Lorentz subgroup and four anti-de Sitter boosts $P_a=:(1/l)\mathscr{K}_{4a}$, with l the de Sitter length. The restriction of μ' to the Lorentz sub-bundle, here the bundle $O(M)$ of orthonormal frames over M, splits according to the Lie algebra structure into the sum of a Lorentz connection $\omega'=\frac{1}{2}\omega^{ab}J_{ab}$ and an $\mathbb{R}^{1,3}$-valued part $\phi'=l\omega^{4a}P_a\equiv\theta^aP_a$, identified with the canonical or soldering form θ on $O(M)$. The restriction to $O(M)$ of the anti-de Sitter curvature two-form, that is, $\Delta'=D^{\mu'}\mu'=\frac{1}{2}F^{AB}\mathscr{K}_{AB}$, has components

$$F^{ab}=\Omega^{ab}+l^{-2}\theta^a\theta^b \tag{7.1}$$

$$F^{4a}=l^{-1}\Theta^a \tag{7.2}$$

where Ω^{ab} is the curvature two-form of the Lorentz frame bundle $O(M)$ calculated from the Lorentz connection ω^{ab}, and Θ^a the torsion two-form of $O(M)$. The kinetic term $\frac{1}{2}C_1 F^{AB*}F_{AB}$ for the anti-de Sitter gauge field in (6.7) then reduces to a gravitational Lagrangian which, besides curvature-squared and torsion-squared terms, also contains the Einstein action with a cosmological term. In fact,

$$\tfrac{1}{2}C_1 F^{AB*}F_{AB} = C_1(\tfrac{1}{2}\Omega^{ab*}\Omega_{ab} - l^{-2}\Theta^{a*}\Theta_a + \tfrac{1}{2}l^{-2}\varepsilon_{abcd}\Omega^{ab}\theta^c\theta^d + \tfrac{1}{4}l^{-4}\varepsilon_{abcd}\theta^a\theta^b\theta^c\theta^d). \tag{7.3}$$

If we put $C_1 = (1/16\pi)(1/g_l^2)$, $g_l = l_p/l$ where l_p is the Planck length, then the Lagrangian (6.7) projected to the base manifold defines an Einstein–Cartan–Yang–Mills theory. The constants C_2, C_3 and C_4 can be identified as the gauge couplings from strong and electroweak interaction, i.e. $C_2 = -1/(2g_s^2)$, $C_3 = -1/(2g^2)$, $C_4 = -1/(2g'^2)$. The other components of the total Lagrangian (6.6) give a further matter Lagrangian for the gravitation theory. Gravitational Lagrangians of type (7.3), where the Einstein Lagrangian is combined with a Lagrangian corresponding to that of a Yang–Mills theory of the Lorentz group, have been considered before [19, 20]. Field equations are obtained from (7.3) by independent variations with respect to θ^a and ω^{ab}. If (7.3) is supplemented with the matter Langrangian $\mathscr{L}_{\mathrm{M}}$, they have the form

$$\tfrac{1}{2}l^{-2}\varepsilon_{abcd}\Omega^{ab}\theta^c + l^{-2}D^*\Theta_d + \tfrac{1}{2}l^{-4}\varepsilon_{abcd}\theta^a\theta^b\theta^c = 8\pi g_l^2 T_d^{(\mathrm{M})} - \tfrac{1}{2}T_d^{(\Omega)} + l^{-2}T_d^{(\Theta)} \tag{7.4}$$

$$l^{-2}\varepsilon_{abcd}\Theta^c\theta^d - 2l^{-2*}\Theta_{[a}\theta_{b]} + D^*\Omega_{ab} = 8\pi g_l^2 S_{ab} \tag{7.5}$$

where the energy momentum and spin angular momentum three-forms of the matter fields are defined as

$$\delta_\theta \mathscr{L}_{\mathrm{M}} = \delta\theta^a T_a^{(\mathrm{M})} \tag{7.6a}$$

$$\delta_\omega \mathscr{L}_{\mathrm{M}} = \tfrac{1}{2}\delta\omega^{ab} S_{ba}. \tag{7.6b}$$

$T_d^{(\Omega)}$ and $T_d^{(\Theta)}$ are the Yang–Mills energy-momentum three-forms associated with the curvature kinetic energy $-\frac{1}{2}\Omega^{ab*}\Omega_{ab}$ and the torsion kinetic energy $-\frac{1}{2}\Theta^{a*}\Theta_a$, respectively [19, 20]. We note that there is no third (redundant) field equation which appears in metric-affine gauge theories [21] by considering the metric tensor of the base manifold M as a third independent geometric potential. In the present formulation, with the gravitational Lagrangian defined on the Lorentz frame bundle, everything is referred to orthonormal frames, and there is no possibility for considering changes of the metric tensor. From the first Bianchi identity $D\Theta^a = \Omega^{ab}\theta_b$ and the field equations, one obtains the Noether identity for the Lorentz symmetry

$$^*DS_{ab} = {}^*(\theta_b T_a^{(\mathrm{M})} - \theta_a T_b^{(\mathrm{M})}) \tag{7.7}$$

where we also used that $(^*D^*)^2\Omega_{ab} = 0$ [17] and the fact that the currents $T_d^{(\Omega)}$ and $T_d^{(\Theta)}$ are symmetric. For zero torsion (7.5) reduces to the Yang–Mills field equation for an $SO(1, 3)$ gauge theory with $8\pi g_l^2$ as a dimensionless coupling. This equation then implies that the left-hand side of (7.7) vanishes such that $\theta_{[a}T_{b]}^{(\mathrm{M})} = 0$, which means that for zero torsion the energy-momentum tensor $T_{ab}^{(\mathrm{M})}$ of the matter fields is symmetric and can be identified as the Belinfante–Rosenfeld energy momentum tensor [22]. In a torsionless vacuum, the field equations have anti-de Sitter space [23] with constant negative curvature $-1/l^2$ as a solution since $T_d^{(\Omega)}$ vanishes for Einstein spaces [24]. For this solution, the cosmological term vanishes if the de Sitter length $l \to \infty$, corresponding to an anti-de Sitter → Poincaré contraction of the spacetime symmetry group. In this limit, the theory in fact reduces to the Stephenson–Yang theory, studied extensively by

Fairchild [24]. With $l \to l_p$ and the torsion constrained to zero, (7.4) reduces to Einstein's field equation as long as the curvature of spacetime does not approach $1/l_p^2$. The vacuum then has an enormous (negative) energy density. However, quantum corrections will generate such a cosmological term even if it was initially absent [25]. Since l_p = constant, the limits considered are determined by the value of g_l^2. In a quantum field theory, the dimensionless coupling g_l^2 of the anti-de Sitter gauge theory is not a constant but evolves, just like the other couplings in the theory, according to the β-functions in the renormalization group equation. Solving these equations is necessary to determine the effective gravitational theory. It will also make a contribution towards solving the problem of the cosmological term in the contemporary universe.

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Appendix

In this appendix we list our choice of conventions for the Pauli, Gell-Mann and Dirac matrices, and introduce further notation.

The Pauli matrices $\sigma_s = \sigma_s^\dagger$, $s = 1, 2, 3$, provide a set of basis operators of the fundamental irrep of $su(2)$:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{A1}$$

The Gell-Mann matrices $\lambda_k = \lambda_k^\dagger$, $k = 1, 2, \ldots, 8$, provide a set of basis operators of the fundamental irrep of $su(3)$:

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_2 = \begin{bmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \lambda_5 = \begin{bmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{bmatrix} \qquad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{A2}$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{bmatrix} \qquad \lambda_8 = (1/\sqrt{3}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

The 4×4 matrices $\sigma_{AB} = \{\sigma_{ab}, \sigma_{4a}\}$, $\sigma^{AB} = \eta^{AC}\eta^{BD}\sigma_{CD}$, $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1, -1)$, $a, b = 0, 1, 2, 3$, defined as

$$\sigma_{ab} =: -\tfrac{1}{4}\mathrm{i}[\gamma_a, \gamma_b] \qquad \sigma_{4a} = \tfrac{1}{2}\gamma_a \tag{A3}$$

provide a set of basis operators of the fundamental spinor irrep of $so(2, 3)$, and $\sigma_{AB}^\dagger = \sigma^{AB}$. The Dirac matrices $\gamma^a = \eta^{ab}\gamma_b$ generating the real Clifford algebra $C(1, 3)$, are represented by

$$\gamma^0=\begin{bmatrix}1_2 & 0_2\\ 0_2 & -1_2\end{bmatrix} \qquad \gamma^s=\begin{bmatrix}0_2 & \sigma_s\\ -\sigma_s & 0_2\end{bmatrix} \qquad s=1,2,3. \tag{A4}$$

We also introduce $\gamma^4=\mathrm{i}I_4$ and the five-vector operator $\{\gamma^A\}=\{\gamma^0,\gamma^1,\gamma^2,\gamma^3,\gamma^4\}$, $\gamma_A=\eta_{AB}\gamma^B$. Finally,

$$\gamma^5=:\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3=\gamma_5=\begin{bmatrix}0_2 & 1_2\\ 1_2 & 0_2\end{bmatrix} \tag{A5}$$

and

$$C=-\mathrm{i}\gamma^0\gamma^2=-\mathrm{i}\begin{bmatrix}0_2 & \sigma_2\\ \sigma_2 & 0_2\end{bmatrix} \tag{A6}$$

is the charge conjugation matrix.

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